

Complex locus of a circle

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(1) It is easy to show that $|z - z_1| = a$, where $z_1 \in \mathbb{C}$, $a \in \mathbb{R}$ form a circle with centre $P_1(z_1)$ and radius a , using an Argand Diagram.

(2) By putting $z = x + yi$ and $z_1 = x_1 + y_1i$, we can transform the equation to well known Cartesian form: $(x - x_1)^2 + (y - y_1)^2 = a^2$. The equation, in fact, is a circle with centre (x_1, y_1) and radius a in the rectangular plane.

(3) Squaring the equation of circle in (1), we get

$$|z - z_1|^2 = a^2 \Rightarrow (z - z_1)(\overline{z - z_1}) = a^2 \Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = a^2 \Rightarrow z\bar{z} - z_1\bar{z} - \bar{z}_1z + z_1\bar{z}_1 = a^2$$

We get another form of circle: $z\bar{z} - z_1\bar{z} - \bar{z}_1z + c = 0$, $a \in \mathbb{C}$, $c \in \mathbb{R}$.

Here $a^2 = |z_1|^2 - c > 0$ in order not to get an imaginary or degenerate circle.

(4) Putting $z = x + yi$, $z_1 = x_1 + y_1i$ in (3) gives back the Cartesian form of the circle.

(5) Putting $z = r(\cos \theta + i \sin \theta)$, $z_1 = \rho(\cos \alpha + i \sin \alpha)$, (ρ, α are constants) in (3):

$$z\bar{z} - z_1\bar{z} - \bar{z}_1z + c = 0 \Rightarrow r^2 - r\rho[\cos(\alpha - \theta) + i \sin(\alpha - \theta)] - r\rho[\cos(\theta - \alpha) + i \sin(\theta - \alpha)] + c = 0$$

we get the polar form of a circle:

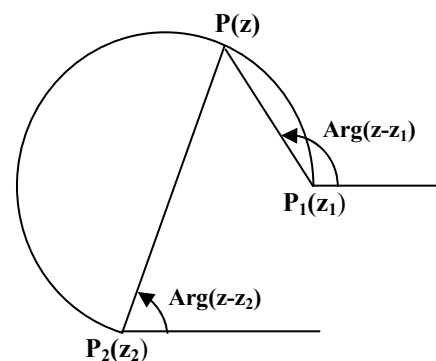
$$r^2 - r\rho \cos(\theta - \alpha) + c = 0, \text{ with centre } = (\rho, \alpha) \text{ and radius } = \sqrt{\rho^2 - c}.$$

(6) $\arg \frac{z - z_1}{z - z_2} = \alpha$, $0 < \alpha < \pi$ gives an arc and **not** a circle.

As in the figure, the locus gives an arc of the circle standing

on the chord with end points z_1 and z_2 such that

$\angle P_1PP_2 = \alpha$ is subtended by the chord at points on the arc, using the \angle s in the same segment theorem.



(7) Putting $z = x + yi$, $z_1 = x_1 + y_1i$, $z_2 = x_2 + y_2i$ in (6), we have:

$$\arg \frac{(x - x_1) + (y - y_1)i}{(x - x_2) + (y - y_2)i} = \alpha \Rightarrow \arg[(x - x_1) + (y - y_1)i] - \arg[(x - x_2) + (y - y_2)i] = \alpha$$

$$\Rightarrow \tan^{-1}\left[\frac{y-y_1}{x-x_1}\right] - \tan^{-1}\left[\frac{y-y_2}{x-x_2}\right] = \alpha \quad \Rightarrow \tan\left\{\tan^{-1}\left[\frac{y-y_1}{x-x_1}\right] - \tan^{-1}\left[\frac{y-y_2}{x-x_2}\right]\right\} = \tan \alpha$$

$$\Rightarrow \frac{\frac{y-y_1}{x-x_1} - \frac{y-y_2}{x-x_2}}{1 + \frac{y-y_1}{x-x_1} \frac{y-y_2}{x-x_2}} = \tan \alpha$$

$$\Rightarrow (x-x_2)(y-y_1) - (x-x_1)(y-y_2) = (\tan \alpha)[(x-x_1)(x-x_2) + (y-y_1)(y-y_2)]$$

(8) The last equation in (7) is a homogenous equation of degree 2, also

(a) coeff. of x^2 -term = coeff. of y^2 term and

(b) there is no xy -term,

It therefore gives a **complete** circle and not an arc.

The "problematic" step in (7) marked by " \Rightarrow " changes the arc into a circle.

(9) The locus of P in (7) represents :

(a) when $\alpha = 0$, the whole line P_1P_2 with the line segment P_1P_2 removed.

(b) when $\alpha = \pi$, the line segment P_1P_2 .

(c) when $0 < \alpha < \pi$, an arc of a circle, terminating at P_1 and P_2 (and excluding these points)

(d) when $\alpha = \pi/2$, a semicircle and the supplementary semicircle is given by $\alpha = 3\pi/2$.

(10) $\arg \frac{z-z_1}{z-z_2} = \alpha$ or $\pi + \alpha$, $0 < \alpha < \pi$ gives a complete circle with P_1 and P_2 removed.

You may investigate the following loci :

(a) $\arg \frac{z-z_1}{z-z_2} = \alpha$ or $\pi - \alpha$, $0 < \alpha < \pi$.

(b) $\arg \frac{z-z_1}{z-z_2} = \pm \alpha$, $0 < \alpha < \pi$.

(11) $\left| \frac{z-z_1}{z-z_2} \right| = k$ where $z_1, z_2 \in \mathbb{C}$, $k > 0$, $k \neq 1$ gives a circle (excluding points $P_1(z_1), P_2(z_2)$)

Note : When $k = 1$, the locus is the perpendicular bisector of the line joining $P_1(z_1)$ and $P_2(z_2)$.

Proof :

$$|z-z_1|^2 = k^2|z-z_2|^2 \Rightarrow (z-z_1)\overline{(z-z_1)} = k^2(z-z_2)\overline{(z-z_2)} \Rightarrow (z-z_1)\overline{(z-z_1)} = k^2(z-z_2)\overline{(z-z_2)}$$

$$\Rightarrow \overline{z}z - \overline{z_1}z - z_1\overline{z} + |z_1|^2 = k^2(\overline{z}z - \overline{z_2}z - z_2\overline{z} + |z_2|^2)$$

$$\Rightarrow (1-k^2)\overline{z}z - (\overline{z_1} - k^2\overline{z_2})z - (z_1 - k^2z_2)\overline{z} + |z_1|^2 - k^2|z_2|^2 = 0$$

$$\Rightarrow z\bar{z} - \left(\frac{\bar{z}_1 - k^2\bar{z}_2}{1-k^2} \right) z - \left(\frac{z_1 - k^2z_2}{1-k^2} \right) \bar{z} + \frac{|z_1|^2 - k^2|z_2|^2}{1-k^2} = 0$$

$$\Rightarrow z\bar{z} - \overline{\left(\frac{z_1 - k^2z_2}{1-k^2} \right)} z - \left(\frac{z_1 - k^2z_2}{1-k^2} \right) \bar{z} + \frac{|z_1|^2 - k^2|z_2|^2}{1-k^2} = 0$$

Comparing this with that given in (3), we get a circle with centre $\frac{z_1 - k^2z_2}{1-k^2}$ and radius a , where

$$a^2 = \left| \frac{z_1 - k^2z_2}{1-k^2} \right|^2 - \frac{|z_1|^2 - k^2|z_2|^2}{1-k^2} \Rightarrow a = k \left| \frac{z_1 - z_2}{1-k^2} \right| \text{ on simplification (exercise)}$$

(12) $\left| \frac{z-z_1}{z-z_2} \right| = k \Rightarrow |z-z_1| = k|z-z_2|$ and if we take $P(z)$ a variable point and $P_1(z_1)$ and $P_2(z_2)$,

we have $P_1P = k P_2P$. This then reduces to a well-known geometry problem :

The Circle of Apollonius: Given two fixed points P_1 and P_2 , the locus of point P such that the ratio of P_1P to P_2P is constant, k , is a circle.

The Circle of Apollonius is not discussed here. Interested readers may consult web-sites such as: <http://jwilson.coe.edu/emt725/Apollonius/Cir.html>

If we know that the locus is a circle, then finding the centre and radius is easier.

As in the diagram, C is the centre and AB is the diameter of the circle.

Then A and B divide P_1P_2 internally and externally :

$$P_1A : AP_2 = k : 1$$

$$P_1B : BP_2 = -k : 1$$

\therefore By section formula:

$$A \text{ represents } \frac{z_1 + kz_2}{1+k}$$

$$B \text{ represents } \frac{z_1 - kz_2}{1-k}$$

$$\Rightarrow \text{The centre of the circle represents } \frac{1}{2} \left(\frac{z_1 + kz_2}{1+k} + \frac{z_1 - kz_2}{1-k} \right) = \frac{z_1 - k^2z_2}{1-k^2}$$

$$\text{and the radius} = |CA| = \left| \frac{z_1 + kz_2}{1+k} - \frac{z_1 - k^2z_2}{1-k^2} \right| = k \left| \frac{z_1 - z_2}{1-k^2} \right|$$

